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NOTES

Edited by **William Adkins**

Two Proofs of Graves's Theorem

Kamal Poorrezaei

1. INTRODUCTION. This note presents two proofs of Graves's theorem on confocal ellipses. The first proof is based on mechanics and the second on the well-known method of drawing an ellipse with two pins and a loop.

Theorem (Graves). *Let Ω and Ψ be confocal ellipses such that Ω is interior to Ψ, and let tangents PA and PB be drawn to Ω from a point P on Ψ (Figure 1). If $\ell(AB)$ denotes the length of the shorter arc of Ω determined by A and B, then $|PA| + |PB| - \ell(AB)$ is constant as P varies over Ψ.*

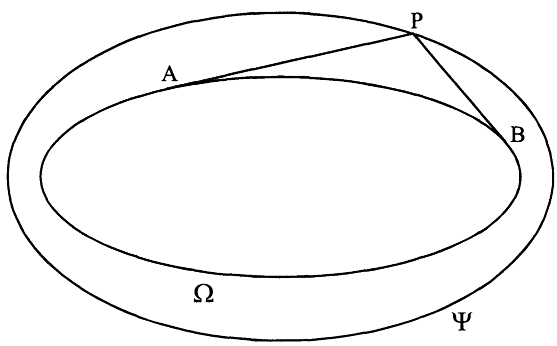

Figure 1.

If $\tilde{\ell}(AB)$ denotes the length of the longer arc AB of Ω, then by adding the length of Ω to the expression $|PA| + |PB| - \tilde{\ell}(AB)$, we see that the quantity $|PA| + |PB| + \tilde{\ell}(AB)$ is also constant as P varies over Ψ.

2. PROOF BY MECHANICS. An isolated system with at least one degree of freedom tends to lower its energy. Deviation from a current state is possible if and only if the system goes to a state of lower energy. Therefore, if an isolated system is in static equilibrium over a continuous range of configurations, its energy must remain constant over this range. In order to apply this principle to the situation in the Graves theorem, we make a mechanical interpretation of the problem.

In Figure 2, suppose that Ω is a solid elliptical plate with a grooved circumference and that Ψ is an elliptical wire, confocal with Ω, on which a small bead P can slide freely. Both Ω and Ψ are fixed to a frame and an elastic loop is strung around Ω and joined to Ψ at the bead P.

Using an elastic loop whose length, when unstretched, is less than the perimeter of Ω ensures that it is always under tension for any position of the bead P. Being taut,

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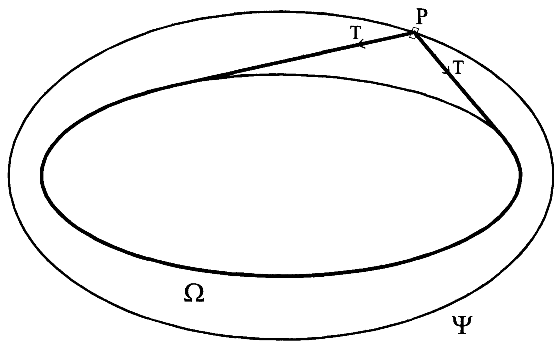

Figure 2.

the loop leaves the groove of Ω and goes toward P along tangents PA and PB. We also stipulate that there be no friction in the system, and thus a tension T is maintained along the elastic loop.

Our goal is to show that the energy of the system remains constant as P moves around Ψ. Since P is acted upon by equal tensions T in the directions of PA and PB, the resultant force on P acts along the bisector of $\angle APB$.

Now the reflection property of an ellipse implies that the normal to Ψ at P bisects the angle F_1PF_2 between its focal radii. Moreover, a lovely theorem of Poncelet **[2]** states that, if tangents PA and PB are drawn to an ellipse from a point outside, then the angles the tangents make with the lines joining P to the foci are equal. Thus we have (Figure 3)

$$\angle APF_1 = \angle F_2PB,$$

from which it follows that the bisectors of angles APB and F_1PF_2 are the same line. Hence the resultant force on the bead P lies along the normal to Ψ, that is, the force acts perpendicularly to Ψ. Thus the system remains in static equilibrium at each position of P, and since the energy of the system is characterized by the length of the elastic loop, the length of the loop must remain constant as P moves around Ψ. Thus $|PA| + |PB| + \widetilde{\ell}(AB)$ is constant, where $\widetilde{\ell}(AB)$ signifies the length of the loop

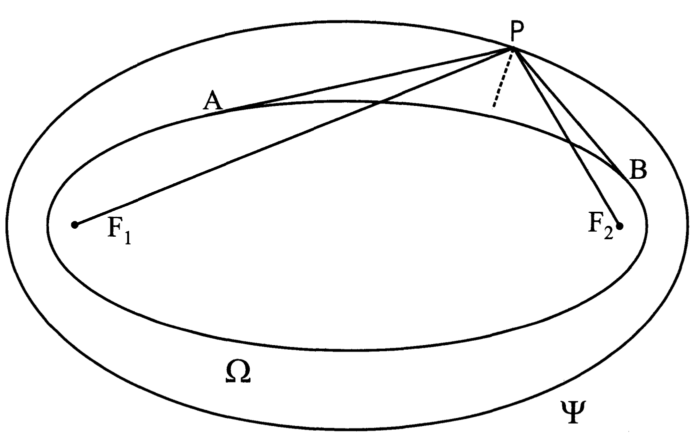

Figure 3.

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that is in contact with Ω. Subtracting the length of Ω, we conclude that the quantity $|PA| + |PB| - \ell(AB)$ is constant as well.

3. A GEOMETRICAL PROOF. Let P be an arbitrary but fixed point on Ψ, and let L_P denote the value of $|PA| + |PB| + \ell(AB)$, as in Figure 1. Also, let L_S denote the value of $|SM| + |SN| + \ell(MN)$ for an arbitrary point S outside Ω, where SM and SN are tangents from S to Ω and MN is the longer arc of Ω determined by these tangents. We show that any curve C through P for which $L_S = L_P$ for all points S of C must be an ellipse that is confocal with Ω, thus making C identical with Ψ. Let an n-gon Ω_n be inscribed in Ω so that its vertices $A_1, A_2, \ldots, A_n$ described in the counterclockwise orientation are equally spaced around Ω, and A_1 is at an apogee of Ω (Figure 4). Thus $\ell(A_1A_2) = \ell(A_2A_3) = \cdots = \ell(A_nA_1)$.

Definition. For the convex n-gon Ω_n the *tangents* to Ω_n from a point S outside it are the rays emanating from S that constrain the viewing angle of Ω_n from S. If SA_i and SA_j are tangents in this sense, then any ray issuing from S and meeting Ω_n lies inside the convex angle A_iSA_j.

Let tangents SA_i and SA_j be drawn to Ω_n from a variable point S in the plane of Ω_n. The vertices A_i and A_j divide the circumference of Ω_n into a part closer to S and a part remote from S. Now let S describe a curve Ψ_n subject to the condition that the sum of the lengths of SA_i, SA_j, and the part of Ω_n that is remote from S is constantly equal to the value L_P that was determined by the initial point P on Ψ:

$$|SA_i| + |SA_j| + \sum_{m=i}^{j} \ell(A_mA_{m+1}) = L_p, \tag{1}$$

where $A_{n+1} = A_1$. Figure 4 shows Ω_6 and its related curve Ψ_6.

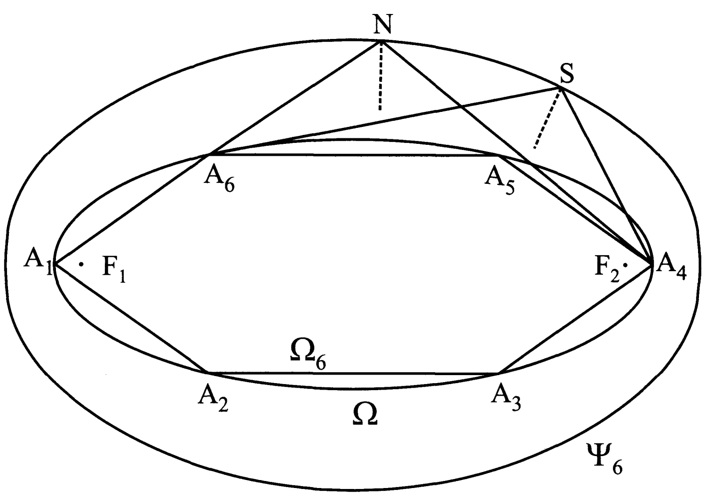

Figure 4.

Now let a pin be placed at each vertex A_i of Ω_n, and let an unstretchable loop of length L_P be strung around them. As shown in Figure 4, keeping the loop taut with a pencil at S, the curve Ψ_n is traced out by starting at P and letting S evolve according to (1) as it moves around Ω_n. Various pairs of vertices of Ω_n take turns as the vertices

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of contact during this process, but so long as the same vertices A_i and A_j remain the vertices of contact, the part of Ω_n that is remote from S also remains unchanged, and therefore the sum $|SA_i| + |SA_j|$ is constant over this segment, implying that Ψ_n is a piecewise elliptical curve.

With A_i and A_j as local foci for a section, SA_i and SA_j are the local radii of S, and the reflection property of ellipses implies that the normal to this elliptical segment is the bisector of $\angle A_i S A_j$. What happens at a point where two such elliptical segments of Ψ_n meet? Consider Figure 4. The point N, being in line with A_6 and A_1, is an endpoint of two segments. Thus as S passes through N in a counterclockwise direction, the vertices of contact (i.e., the foci of the segment) switch from A_6 and A_4 to A_1 and A_4. Thus on one side of N the normal is the bisector of $\angle A_6 N A_4$ and on the other it is the bisector of $\angle A_1 N A_4$. At N itself, where the angles $A_6 N A_4$ and $A_1 N A_4$ coincide, the elliptical segments have a common normal. This means a common tangent, and it follows that Ψ_n is actually a smooth curve. Moreover, we have that the normal to Ψ_n at any point S is the bisector of the angle between the tangents from S to Ω_n.

As n increases, Ω_n approaches Ω in the limit, in which case Ψ_n approaches a limit Ψ_∞. Also, in the limit, tangents to Ω_n become tangents to Ω. Thus the bisector of the angle between the tangents to Ω from a point S on Ψ_∞ is the normal to Ψ_∞ at S. Now invoking Poncelet's theorem again, we infer that $\angle A_i S F_1 = \angle F_2 S A_j$, in which case the bisector of $\angle A_i S A_j$ coincides with the bisector of $\angle F_1 S F_2$. We conclude that Ψ_∞ has two points F_1 and F_2 inside it with the property that the normal at each point S on Ψ_∞ is the bisector of $F_1 S F_2$. In the following lemma we show that this implies that Ψ_∞ is an ellipse with foci F_1 and F_2. That is, Ψ_∞ is an ellipse that is confocal with Ω. Since Ψ_∞ clearly goes through the fixed point P that was chosen on Ψ, it follows that $\Psi_\infty = \Psi$. Finally, in the limit, the condition (1) that governs the evolution of Ψ_n becomes the alternative sum in Graves's theorem, so the proof is complete.

Lemma. *If Ψ is a smooth, closed, convex curve in R^2 having inside it points F_1 and F_2 such that the normal to Ψ at any point S bisects $\angle F_1 S F_2$, then Ψ is an ellipse with foci F_1 and F_2.*

Proof. We wish to show that $|SF_1| + |SF_2|$ is constant for S on Ψ. Suppose to the contrary that $|SF_1| + |SF_2|$ is not constant but attains a maximum value M at some

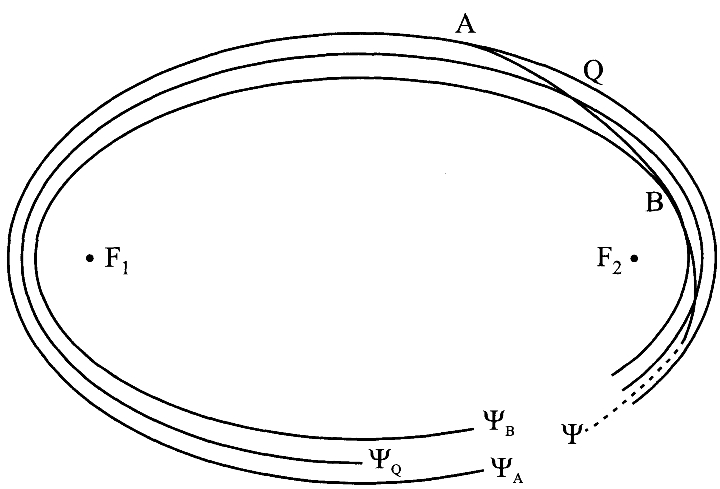

Figure 5.

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point A and a minimum value m at some point B, where $m < M$. Thus for any point Q of Ψ different from A and B, we have

$$BF_1 + BF_2 \leq QF_1 + QF_2 \leq AF_1 + AF_2$$

and at least one of the inequalities is strict. Now let three ellipses Ψ_A, Ψ_B, and Ψ_Q with foci F_1 and F_2 be drawn to pass through A, B, and Q, respectively (Figure 5). Since Ψ has the same reflection property as the ellipses, it must be tangent to them at the contact points A, B, and Q. However, the shorter arc AB of Ψ runs from a point B inside Ψ_Q to a point A outside it. Since the curve Ψ is convex, it could not then be tangent to Ψ_Q. This contradiction leads us to the conclusion that $m = M$, and thus all four curves must coincide. ■

REFERENCES

1. E. Goursat, *A Course in Mathematical Analysis*, vol. 1, 5th ed., Dover, New York, 1959.
2. H. F. Baker, *Principles of Geometry*, vol. 2, Frederick-Ungar, New York, 1968.

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A Characterization of the Unit Sphere

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Let $\gamma(s)$ be a unit speed curve in $\mathbb{R}^{n+1}$, so $|\gamma'(s)| = 1$ for all s. Then the *curvature* $\kappa(s)$ of $\gamma(s)$ is the length of the acceleration vector $\gamma''(s)$. If $\gamma(s)$ lies on the unit sphere

$$S^n = \{x \in \mathbb{R}^{n+1} \mid x \cdot x = 1\},$$

then the curvature of γ is everywhere greater than or equal to 1: differentiating the relation $\gamma(s) \cdot \gamma(s) = 1$ twice yields $\gamma''(s) \cdot \gamma(s) = -1$, and the Schwartz inequality gives $\kappa(s) = |\gamma''(s)| \geq 1$. Of course, the spheres of radius not exceeding 1 also have this curvature property. Thus it is natural to ask what conditions should be added to characterize the unit sphere in $\mathbb{R}^{n+1}$.

We prove the following:

Theorem. *Suppose that M is a closed hypersurface in $\mathbb{R}^{n+1}$ that satisfies the following two conditions:*

(C_1) *every curve on M has curvature ≥ 1;*

(C_2) *on M there exists a curve γ_0 of length π with constant curvature 1.*

Then M is the unit sphere.

Let U be a given unit vector field normal to the hypersurface M. If p is a point of M, $T_p(M)$ denotes the tangent space of M at p and the so-called shape operator

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